

Optimal Placement of Distributed Energy Storage in Power Networks

Christos Thrampoulidis, *Student Member, IEEE*, Subhonmesh Bose, *Student Member, IEEE*, and Babak Hassibi *Fellow, IEEE*.

Abstract

Large-scale storage is a promising emerging technology to realize a reliable smart-grid since it can enhance sustainability, reliability and asset utilization. At fast time-scales, storage can be used to mitigate fluctuations and time-variations on renewable generation and demand. On slower time scales, it can be used for load shifting and reducing the generation costs. This paper deals with the latter case and studies the following important decision problem: Given a network and an available storage budget, how should we optimally place, size and control the energy storage units across the network. When the generation cost is a nondecreasing convex function, our main result states that it is always optimal to allocate zero storage capacity at generator buses that connect to the rest of the power grid via single links, regardless of demand profiles and other network parameters. This provides a sound investment strategy for network planners, especially for distribution systems and isolated transmission networks. For specific network topologies, we also characterize the dependence of the optimal production cost on the available storage resources, generation capacities and flow constraints.

I. INTRODUCTION

Emerging energy storage technologies in the electricity grid can be potentially used for various services, e.g., to enhance sustainability, reliability, efficiency and asset utilization [1]–[4]. The economic value [5]–[7] and national impact [8], [9] of these services has been emphasized in various recent studies. Over the last decade or so, these technologies have shown remarkable

Emails: (cthrampo, bose, hassibi)@caltech.edu. This work was supported in part by NSF under grants CCF-0729203, CNS-0932428 and CCF-1018927, NetSE grant CNS 0911041, Office of Naval Research MURI grant N00014-08-1-0747, Caltech Lee Center for Adv. Net., ARPA-E grant de-ar0000226, Southern Cali. Edison, Nat. Sc. Council of Taiwan, R.O.C. grant NSC 101-3113-P-008-001, Resnick Inst., Okawa Foundation and Andreas Mentzelopoulos Scholarships for the Univ. of Patras.

technical improvements and their cost of production has dropped significantly [10]–[12]. Also, the obvious challenges in integrating intermittent renewable generation has spurred the interest in large-scale storage to deal with their variability [13]–[16]. In summary, the authors in [8] identify the use of energy storage as *“critical to achieving national energy policy objectives and creating a modern and secure electric grid system.”* For this promising new technology, we study an efficient investment strategy of placing and sizing such units in a network to minimize generation cost.

Applications of energy storage can be classified according to time scale of operation [3], [5], [17]. At fast-time scales (seconds to minutes), storage is mainly used to mitigate variability of renewable generation and demand and reduce the role of costly ancillary services to balance demand and supply, e.g., [18]–[21]. On a slower time scale (over hours), bulk storage devices aim at load shifting, i.e., generate when it is cheap and supply the residual load using storage dynamics [5], [10]. In the present work, we only deal with the latter. Our results are therefore complementary to those of [18]–[21]. The short-time scale strategies developed there can be implemented “on top” of the long-term strategies developed here.

Now, we provide a brief overview of the relevant literature. Optimal control policy for storage units has been extensively studied. While the authors in [20], [22] examine the control of a single storage device without a network, [21], [23], [24] explicitly model the role of the networks in the operation of distributed storage resources. The engineering constraints are designed based on one of the two popular power flow models [25]–[27], namely, AC power flow and DC power flow. Storage resources at each node in the network are assumed to be known *a priori* in these settings.

For the emerging smart grid technology, the investment decision problem of selecting, sizing and placement of distributed energy resources is gaining importance. As noted in [28], it is critical to *“assess the technical and economic attributes of energy storage specifically reflecting the operational demands and opportunities presented by the smart grid environment. Without the ability to analyze network features of transmission and distribution, storage system technologies and their efficiencies, along with their cost benefits for various value streams, there is no ability for the utility to make comparative business decisions that will enable the optimal siting of energy storage.”* Several researchers have analyzed aspects of this problem, e.g., [29], [30] using purely economic arguments, without explicitly considering the network constraints of the

physical system. Authors in [20], [31] have looked at optimal sizing of storage devices in single-bus power systems, while Kanoria et al. [21] compute the effect of sizing of distributed storage resources on generation cost for specific networks. Gayme et. al [23] study a similar problem in IEEE benchmark systems [32]. Recently, a more general framework to study the optimal storage placement problem in generic networks has been formulated and studied through simulations in [33], [34] using tools similar to optimal power flow (OPF) . The resulting problem is non-convex and hence Sjödin et al. in [33] use the approximate DC-OPF [26], [27] while Bose et al. in [34] use the relaxation of AC-OPF [35]–[38] based on semidefinite programming [39], [40].

In this paper, we consider the investment decision problem of how much storage to place on each node in a network given an available storage budget. As a by-product, we also derive the optimal control policy for the storage units. Our main result shows that *when minimizing a convex and nondecreasing generation cost with any fixed available storage budget, there always exists an optimal storage allocation that assigns zero storage at those generator nodes that connect via single lines to the network, for arbitrary demand profiles and other network parameters*. It suggests that in most distribution networks and isolated transmission networks, it is always optimal to allocate the entire available storage budget among demand buses. Also, this work provides an analytic justification to the conjecture that the optimal allocation of storage resources mainly depends on the network structure as opposed to the total available storage budget as noted through simulations in [23], [33], [34]. For generator buses with multiple line connections, however, we show that the optimal storage placement may not in general place zero storage capacity at such nodes. Furthermore, we study the dependence of optimal production cost on the available storage budget, line flow and generation capacities for specific network topologies. The physical network has been modeled using DC power flow. Preliminary versions of our result have appeared in [41] and [42].

The paper is organized as follows. We formulate the optimal storage placement problem in Section II. The main result is proven and discussed in Section III. More detailed analysis of the problem on specific network topologies is discussed in Section IV. Concluding remarks and directions for future work are outlined in Section V. Technical details appear in the Appendix.

II. PROBLEM FORMULATION

Consider a power network that is defined by an undirected connected graph \mathcal{G} on n nodes (or buses) $\mathcal{N} = \{1, 2, \dots, n\}$. For two nodes k and l in \mathcal{N} , let $k \sim l$ denote that k is connected to l in \mathcal{G} by a transmission line. For the power flow model, we use the linearized DC approximation; see [26] for a detailed survey. In this approximation, the network is assumed to be lossless, the voltage magnitudes are assumed to be at their nominal values at all buses and the voltage phase angle differences are small.

Time is discrete and is indexed by t . Now consider the following notation.

- $d_k(t)$ is the known real power demand at bus $k \in \mathcal{N}$ at time t . Demand profiles often show diurnal variations [43], i.e., they exhibit cyclic behavior. Let T time-steps denote the cycle length of the variation. In particular, for all $k \in \mathcal{N}$, $t \geq 0$, assume

$$d_k(t + T) = d_k(t).$$

- $g_k(t)$ is the real power generation at bus $k \in \mathcal{N}$ at time t and it satisfies

$$0 \leq g_k(t) \leq \bar{g}_k, \quad (1)$$

where, \bar{g}_k is the generation capacity at bus k .

- $c_k(g_k)$ denotes the cost of generating power g_k at bus $k \in \mathcal{N}$. The cost of generation is assumed to be independent of time t and depends only on the generation technology at bus k . Also, suppose that the function $c_k : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is non-decreasing, differentiable and convex. A cost function commonly found in the literature [23], [25], [37], [38] that satisfies the above assumptions is:

$$c_k(g_k) = \gamma_{k,2}g_k^2 + \gamma_{k,1}g_k + \gamma_{k,0},$$

where $\gamma_{k,2}, \gamma_{k,1}, \gamma_{k,0}$ are known nonnegative coefficients.

- $\theta_k(t)$ is the voltage phase angle at bus $k \in \mathcal{N}$ at time t .
- For two nodes $k \sim l$, let $p_{kl}(t)$ be the power flowing from bus k to bus l at time t . It satisfies

$$p_{kl}(t) = y_{kl} [\theta_k(t) - \theta_l(t)], \quad (2)$$

where y_{kl} is the admittance of the line joining buses k and l . Also, the power delivered over this line is limited by thermal effects and stability constraints and hence, we have

$$|p_{kl}(t)| \leq f_{kl}, \quad (3)$$

where f_{kl} is the capacity of the corresponding line.

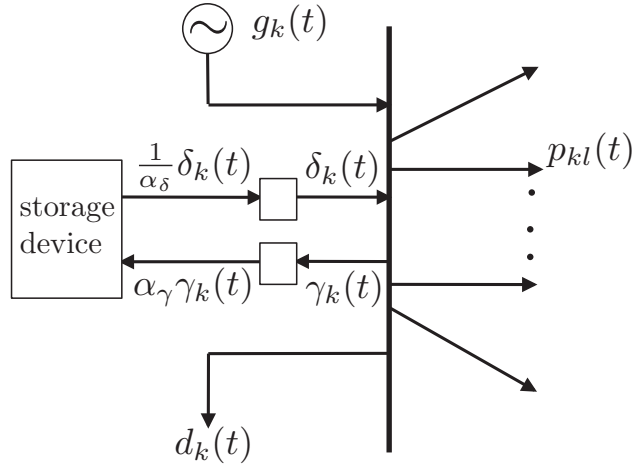


Fig. 1: Power balance at node $k \in \mathcal{N}$.

- $\gamma_k(t)$ and $\delta_k(t)$ are the average charging and discharging powers of the storage unit at bus $k \in \mathcal{N}$ at time t , respectively. The energy transacted over a time-step is converted to power units by dividing it by the length of the time-step. This transformation conveniently allows us to formulate the problem in units of power [34]. Let $0 < \alpha_\gamma, \alpha_\delta \leq 1$ denote the charging and discharging efficiencies, respectively of the storage technology used, i.e., the power flowing in and out of the storage device at node $k \in \mathcal{N}$ at time t is $\alpha_\gamma \gamma_k(t)$ and $\frac{1}{\alpha_\delta} \delta_k(t)$, respectively [20], [44]. The roundtrip efficiency of this storage technology is $\alpha = \alpha_\gamma \alpha_\delta \leq 1$.
- $s_k(t)$ denotes the storage level at node $k \in \mathcal{N}$ at time t and s_k^0 is the storage level at node k at time $t = 0$. From the definitions above, we have that

$$s_k(t) = s_k^0 + \sum_{\tau=1}^t \left(\alpha_\gamma \gamma_k(\tau) - \frac{1}{\alpha_\delta} \delta_k(\tau) \right). \quad (4)$$

For each $k \in \mathcal{N}$, assume $s_k^0 = 0$, so that the storage units are empty at installation time.

- $b_k \geq 0$ is the storage capacity at bus k . Thus, $s_k(t)$ for all t satisfies the following:

$$0 \leq s_k(t) \leq b_k. \quad (5)$$

- h is the available storage budget and denotes the total amount of storage capacity that can be installed in the network. Our optimization algorithm decides the allocation of storage capacity b_k at each node $k \in \mathcal{N}$ and thus, we have

$$\sum_{k \in \mathcal{N}} b_k \leq h. \quad (6)$$

- The charging and discharging rates of each storage device are assumed to be upper-bounded by ramp limits. Suppose these limits are proportional to the capacity of the corresponding device, i.e., for all $k \in \mathcal{N}$,

$$0 \leq \gamma_k(t) \leq \epsilon_\gamma b_k, \quad (7a)$$

$$0 \leq \delta_k(t) \leq \epsilon_\delta b_k, \quad (7b)$$

where $\epsilon_\gamma \in (0, \frac{1}{\alpha_\gamma}]$ and $\epsilon_\delta \in (0, \alpha_\delta]$ are fixed constants.

Balancing power that flows in and out of bus $k \in \mathcal{N}$ at time t , as shown in Figure 1, we have:

$$g_k(t) - d_k(t) - \gamma_k(t) + \delta_k(t) = \sum_{l \sim k} p_{kl}(t). \quad (8)$$

Also, optimally placing storage over an infinite horizon is equivalent to solving this problem over a single cycle, provided the state of the storage levels at the end of a cycle is the same as its initial condition [34]. Thus, for each $k \in \mathcal{N}$, we have

$$\sum_{t=1}^T \left(\alpha_\gamma \gamma_k(t) - \frac{1}{\alpha_\delta} \delta_k(t) \right) = 0. \quad (9)$$

For convenience, denote $[T] := \{1, 2, \dots, T\}$. Using the above notation, we define the following optimization problem.

Storage placement problem P :

$$\begin{aligned}
& \text{minimize} && \sum_{k \in \mathcal{N}} \sum_{t=1}^T c_k(g_k(t)) \\
& \text{over} && (g_k(t), \gamma_k(t), \delta_k(t), \theta_k(t), p_{kl}(t), b_k), \\
& && k \in \mathcal{N}, \quad k \sim l, \quad t \in [T], \\
& \text{subject to} && (1), (2), (3), (4), (5), (6), (7), (8), (9),
\end{aligned}$$

where, (1) represents generation constraints, (2),(3) represent power flow constraints, (5),(6),(7),(9) represent the constraints imposed on the charging/discharging control policy of the energy storage devices, (8) represents the power balance constraints at each bus of the network and (6) represents the constraint on the sum of the capacities of all storage devices being no greater than the available storage budget.

Restrict attention to network topologies where each bus either has generation or load but not both. Partition the set of buses \mathcal{N} into two groups \mathcal{N}_G and \mathcal{N}_D where they represent the generation-only and load-only buses respectively and assume \mathcal{N}_G and \mathcal{N}_D are non-empty. For any subset \mathcal{K} of \mathcal{N}_G , define the following optimization problem.

Restricted storage placement problem $\Pi^{\mathcal{K}}$:

$$\begin{aligned}
& \text{minimize} && \sum_{k \in \mathcal{N}} \sum_{t=1}^T c_k(g_k(t)) \\
& \text{over} && (g_k(t), \gamma_k(t), \delta_k(t), \theta_k(t), p_{kl}(t), b_k), \\
& && k \in \mathcal{N}, \quad k \sim l, \quad t \in [T], \\
& \text{subject to} && (1), (2), (3), (4), (5), (6), (7), (8), (9), \\
& && b_i = 0, \quad i \in \mathcal{K}.
\end{aligned}$$

Problem $\Pi^{\mathcal{K}}$ corresponds to placing no storage at the (generation) buses of the network in subset \mathcal{K} . We study the relation between the problems P and $\Pi^{\mathcal{K}}$ in the rest of the paper.

We say bus $k \in \mathcal{N}$ has a *single connection* if it has exactly one neighboring node $l \sim k$. Similarly, a bus $k \in \mathcal{N}$ has *multiple connections* if it has more than one neighboring node in \mathcal{G} . We illustrate the notation using the network in Figure 2. $\mathcal{N}_G = \{1, 2, 7\}$ and $\mathcal{N}_D = \{3, 4, 5, 6\}$. Buses 1 and 2 have single connections and all other buses in the network have

multiple connections.

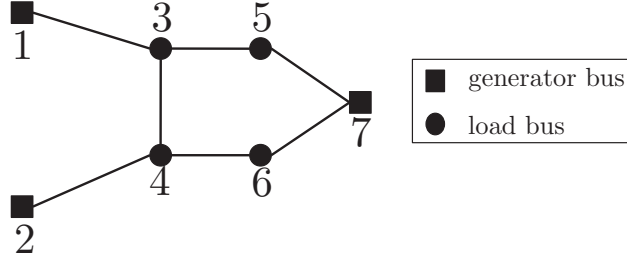


Fig. 2: A sample network.

III. MAIN RESULT

For a subset $\mathcal{K} \subseteq \mathcal{N}_G$, let p_* and $\pi_*^{\mathcal{K}}$ be the optimal values for problems P and $\Pi^{\mathcal{K}}$, respectively. Now, we are ready to present the main result of this paper.

Theorem 1. *Suppose $\mathcal{K} \subseteq \mathcal{N}_G$ and each node $i \in \mathcal{K}$ has a single connection. If P is feasible, then $\Pi^{\mathcal{K}}$ is feasible and $p_* = \pi_*^{\mathcal{K}}$.*

The theorem states that, for any available storage budget, there *always* exists an optimal allocation of storage capacities that places *no* storage at any subset of generation buses with single connections. The result holds, regardless of the demand profiles and other network parameters, such as line flow constraints and impedances of transmission lines. In our model, the demand profiles of the load buses are deterministic. However, since Theorem 1 holds for arbitrary demand profiles, our result extends to the case where the load is stochastic.

Note that we restrict attention to generator buses in \mathcal{K} that have single connections only. This is applicable in many practical scenarios, as discussed in Section III-B. The result is not true, in general, if \mathcal{K} includes generator buses with multiple connections. We provide an example in Section III-C to illustrate this.

A. Proof of Theorem 1

We only prove for the case where the round-trip efficiency is $\alpha < 1$, but the result holds for $\alpha = 1$ as well. Assume P is feasible throughout. For any variable z in problem P , let z^* be the value of the corresponding variable at the optimum. In our proof, we use the following technical result.

Lemma 1. Suppose $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is convex and differentiable. Then, for any $x_1 < x_2$ and $0 \leq \eta \leq (x_2 - x_1)$:

$$\phi(x_1 + \eta) + \phi(x_2 - \eta) \leq \phi(x_1) + \phi(x_2).$$

Proof: We only prove the result for $0 < \eta \leq (x_2 - x_1)/2$. The other cases can be shown similarly. The function $\phi(\cdot)$ is continuous and differentiable. From mean-value Theorem [39], it follows that there exists $\xi_1 \in (x_1, x_1 + \eta)$ and $\xi_2 \in (x_2 - \eta, x_2)$, such that $[\phi(x_1 + \eta) - \phi(x_1)] - [\phi(x_2) - \phi(x_2 - \eta)] = \phi'(\xi_1)\eta - \phi'(\xi_2)\eta$. Also, $\xi_1 < \xi_2$ and hence $\phi'(\xi_1) \leq \phi'(\xi_2)$ since ϕ is convex. This completes the proof. ■

Consider node $i \in \mathcal{K}$ and $j \sim i$. Node j is uniquely defined as i has a single connection. It can be shown that problem P , in general, has multiple optima. In the following result, we characterize only a subset of these optima.

Lemma 2. There exists an optimal solution of P such that for all $t \in [T]$ and all $i \in \mathcal{K}, j \sim i$,

(a) $g_i^*(t)\gamma_i^*(t)\delta_i^*(t) = 0$,

(b) $g_i^*(t) \leq f_{ij}$.

The first part of Lemma 2 essentially says that for some optimum solution of P , the storage units should not charge and discharge at the same time step if there is positive generation at the same bus at that time step. This is expected since the round-trip efficiency of the storage devices $\alpha = \alpha_\gamma \alpha_\delta$ is less than one and since the generation cost is a nondecreasing function. The second part can be interpreted as follows. Power that flows from bus i to bus j at each $t \in [T]$ is $p_{ij}(t) = g_i(t) - \gamma_i(t) + \delta_i(t)$ and we have $p_{ij}(t) \leq f_{ij}$. But Lemma 2(b) states that there exists an optimum for which, $g_i^*(t), t \in [T]$ itself defines a feasible flow over this line.

Proof: The feasible set of problem P is a bounded ¹ polytope and the objective function is a continuous convex function. Hence the set of the optima of P is a convex and compact set [39]. Now, with every point in the set of optimal solutions of P , consider the function $\sum_{i \in \mathcal{K}, t \in [T]} (\gamma_i(t) + \delta_i(t))$. This is a linear and hence continuous function on the compact set of optima of P and hence attains a minimum. Consider the optimum of P where this minimum is

¹ Without loss of generality, let bus 1 be the slack bus and hence $\theta_1(t) = 0$ for all $t \in [T]$. Boundedness of the set of feasible solutions of P then follows from the relations in (1), (2), (3), (6) and (7),.

attained. We show that for this optimum, $g_i^*(t)\gamma_i^*(t)\delta_i^*(t) = 0$ and $g_i^*(t) \leq f_{ij}$ for all $t \in [T]$ and $i \in \mathcal{K}, j \sim i$.

(a) Suppose, on the contrary, we have $g_i^*(t_0) > 0$, $\gamma_i^*(t_0) > 0$ and $\delta_i^*(t_0) > 0$ for some $t_0 \in [T]$.

Define

$$\Delta g' := \min \left\{ (1 - \alpha)\gamma_i^*(t_0), \frac{1 - \alpha}{\alpha}\delta_i^*(t_0), g_i^*(t_0) \right\}.$$

Note that $\Delta g' > 0$. Now, for bus i , construct modified generation, charging and discharging profiles $\tilde{g}_i(t), \tilde{\delta}_i(t), \tilde{\gamma}_i(t), t \in [T]$ that differ from $g_i^*(t), \delta_i^*(t), \gamma_i^*(t)$ only at t_0 as follows:

$$\begin{aligned} \tilde{g}_i(t_0) &:= g_i^*(t_0) - \Delta g', \\ \tilde{\gamma}_i(t_0) &:= \gamma_i^*(t_0) - \frac{1}{1 - \alpha}\Delta g', \\ \tilde{\delta}_i(t_0) &:= \delta_i^*(t_0) - \frac{\alpha}{1 - \alpha}\Delta g'. \end{aligned}$$

Note that, for all $t \in [T]$, the storage level $s_i(t)$ and the power $p_{ij}(t)$ flowing from bus i to bus j remain unchanged throughout. It can be checked that the modified profiles define a feasible point of P . Since $c_i(\cdot)$ is non-decreasing, we have $c_i(\tilde{g}_i(t_0)) \leq c_i(g_i^*(t_0))$ and hence the additivity of the objective in P over i and t implies that this feasible point has an objective function value of at most p_* . It follows that this feasible point defines an optimal point of P . However, we have $\tilde{\gamma}_i(t_0) + \tilde{\delta}_i(t_0) < \gamma_i^*(t_0) + \delta_i^*(t_0)$ and thus, this optimum of P has a strictly lower $\sum_{i \in \mathcal{K}, t \in [T]} (\gamma_i(t) + \delta_i(t))$, contradicting our hypothesis. This completes the proof of $g_i^*(t_0)\gamma_i^*(t_0)\delta_i^*(t_0) = 0$.

(b) If $g_i^*(t) = 0$ for all $t \in [T]$, then $g_i^*(t) \leq f_{ij}$ clearly holds. Henceforth, assume $\max_{t \in [T]} g_i^*(t) > 0$, and consider any $t_0 \in [T]$, such that $g_i^*(t_0) = \max_{t \in [T]} g_i^*(t)$.

If $\gamma_i^*(t_0) = 0$, then,

$$\begin{aligned} \max_{t \in [T]} g_i^*(t) &= g_i^*(t_0) \\ &= \underbrace{p_{ij}^*(t_0)}_{\leq f_{ij}} + \underbrace{\gamma_i^*(t_0)}_{=0} - \underbrace{\delta_i^*(t_0)}_{\geq 0} \\ &\leq f_{ij}. \end{aligned} \tag{10}$$

and Lemma 2(b) holds.

Suppose now that $\gamma_i^*(t_0) > 0$ and hence $\delta_i^*(t_0) = 0$ from Lemma 2(a). First, we show that the storage device discharges at some point after t_0 .

$$s_i^*(t_0) = \underbrace{s_i^*(t_0 - 1)}_{\geq 0} + \underbrace{\alpha_\gamma \gamma_i^*(t_0)}_{> 0} > 0.$$

We also have $s_i^*(T) = s_i^0 = 0$ by hypothesis. Thus the storage device at node i needs to discharge in $[t_0 + 1, T]$ and hence $\alpha_\gamma \gamma_i^*(t) - \frac{1}{\alpha_\delta} \delta_i^*(t) < 0$ for some $t \in [t_0 + 1, T]$. Let t_1 be the first time instant after t_0 when the storage device at bus i is discharged, i.e.

$$t_1 := \min \left\{ t \in [t_0 + 1, T] \mid \alpha_\gamma \gamma_i^*(t) - \frac{1}{\alpha_\delta} \delta_i^*(t) < 0 \right\}. \quad (11)$$

Thus, $\delta_i^*(t_1) > 0$. Define

$$\Delta g := \min \left\{ \gamma_i^*(t_0), \frac{1}{\alpha} \delta_i^*(t_1), g_i^*(t_0) \right\}. \quad (12)$$

Then $\Delta g > 0$. Now, consider the case where:

$$g_i^*(t_1) > 0, \quad \text{and} \quad g_i^*(t_0) \leq g_i^*(t_1) + \alpha \Delta g. \quad (13)$$

Since $g_i^*(t_1) > 0$ and $\delta_i^*(t_1) > 0$, then $\gamma_i^*(t_1) = 0$, by Lemma 2(a). In that case, $g_i^*(t_1) + \delta_i^*(t_1) = p_{ij}^*(t_1)$ is the power that flows from bus i to bus j at time t_1 . Combining (12) and (13), we have

$$\begin{aligned} \max_{t \in [T]} g_i^*(t) &= g_i^*(t_0) \\ &\leq g_i^*(t_1) + \alpha \Delta g \\ &\leq g_i^*(t_1) + \delta_i^*(t_1) \\ &= p_{ij}^*(t_1) \leq f_{ij}. \end{aligned}$$

Hence, Lemma 2(b) holds when (13) is satisfied. Next, we show that if (13) does not hold, then we can construct an optimum of P with a lower $\sum_{i \in \mathcal{K}, t \in [T]} (\gamma_i(t) + \delta_i(t))$ and this contradicts our hypothesis.

Suppose (13) does not hold. If $g_i^*(t_1) = 0$, then we have

$$g_i^*(t_0) \geq \Delta g > \alpha \Delta g = g_i^*(t_1) + \alpha \Delta g.$$

Thus, it suffices to only consider the following case:

$$g_i^*(t_0) > g_i^*(t_1) + \alpha\Delta g. \quad (14)$$

Construct the modified generation, charging and discharging profiles at node i , $\tilde{g}_i(t)$, $\tilde{\delta}_i(t)$, $\tilde{\gamma}_i(t)$ using (12), that differ from $g_i^*(t)$, $\delta_i^*(t)$, $\gamma_i^*(t)$ only at t_0 and t_1 as follows:

$$\begin{aligned} \tilde{g}_i(t_0) &= g_i^*(t_0) - \Delta g, & \tilde{g}_i(t_1) &= g_i^*(t_1) + \alpha\Delta g, \\ \tilde{\gamma}_i(t_0) &= \gamma_i^*(t_0) - \Delta g, & \tilde{\gamma}_i(t_1) &= \gamma_i^*(t_1), \\ \tilde{\delta}_i(t_0) &= \delta_i^*(t_0) = 0, & \tilde{\delta}_i(t_1) &= \delta_i^*(t_1) - \alpha\Delta g. \end{aligned}$$

Also, define the modified storage level $\tilde{s}_i(t)$ using $\tilde{\gamma}_i(t)$ and $\tilde{\delta}_i(t)$. To provide intuition to the above modification, we essentially generate and store less at time t_0 by an amount Δg . This means at a future time t_1 , we can discharge $\alpha\Delta g$ less from the storage device and hence have to generate $\alpha\Delta g$ more to compensate. To check feasibility, it follows from (12), that for $t = t_0, t_1$, we have

$$\begin{aligned} 0 &\leq \tilde{g}_i(t) \leq \overline{g}_i, \\ 0 &\leq \tilde{\gamma}_i(t) \leq \epsilon_\gamma b_i^*, \\ 0 &\leq \tilde{\delta}_i(t) \leq \epsilon_\delta b_i^*. \end{aligned}$$

Also, the line flows $p_{ij}(t)$ remain unchanged. For the storage levels, it can be checked that the following holds:

$$\begin{aligned} 0 \leq s_i^*(t_0 - 1) \leq \tilde{s}_i(t) \leq s_i^*(t) \leq b_i^*, \text{ for } t \in [t_0, t_1 - 1], \\ \tilde{s}_i(t) = s_i^*(t), \text{ otherwise.} \end{aligned}$$

This proves that the modified profiles define a feasible point for P . The cost satisfies

$$\begin{aligned} c_i(\tilde{g}_i(t_0)) + c_i(\tilde{g}_i(t_1)) \\ \leq c_i(g_i^*(t_0) - \alpha\Delta g) + c_i(g_i^*(t_1) + \alpha\Delta g) \end{aligned} \quad (15a)$$

$$\leq c_i(g_i^*(t_0)) + c_i(g_i^*(t_1)). \quad (15b)$$

Equation (15a) follows from the non-decreasing nature of $c_i(\cdot)$ and equation (15b) follows from using (14) and Lemma 1. Thus the modified profiles $\tilde{g}_i(t), \tilde{\delta}_i(t), \tilde{\gamma}_i(t)$ define a feasible point of P with a cost at most p_* and, hence, are optimal for P . However, we also have

$$\begin{aligned} & \tilde{\gamma}_i(t_0) + \tilde{\gamma}_i(t_1) + \tilde{\delta}_i(t_0) + \tilde{\delta}_i(t_1) \\ &= \gamma_i^*(t_0) + \gamma_i^*(t_1) + \delta_i^*(t_0) + \delta_i^*(t_1) - \underbrace{(1 + \alpha)\Delta g}_{>0}. \end{aligned}$$

Thus, the modified profiles define an optimum of P with a lower $\sum_{i \in \mathcal{K}, t \in [T]} (\gamma_i(t) + \delta_i(t))$. This is a contradiction and completes the proof of the Lemma. ■

To prove Theorem 1, consider the optimal solution of P that satisfies Lemma 2(b). For all $i \in \mathcal{K}$, $g_i^*(t)$ itself defines a feasible flow over the line joining buses i and j , where j is the unique neighboring node of i . Now the proof idea is as follows. For $i \in \mathcal{K}$, transfer all storage capacities b_i^* and the associated charging/ discharging profiles $(\gamma_i^*(t), \delta_i^*(t))$, to the neighboring node j . In particular, consider the point $\left(g_k^*(t), \hat{\gamma}_k(t), \hat{\delta}_k(t), \hat{\theta}_k(t), \hat{p}_{kl}(t), \hat{b}_k, k \in \mathcal{N}, k \sim l, t \in [T]\right)$ defined as follows.

$$\hat{\gamma}_i(t) = 0, \quad \hat{\gamma}_j(t) = \gamma_i^*(t) + \gamma_j^*(t),$$

$$\hat{\gamma}_k(t) = \gamma_k^*(t), \quad k \in \mathcal{N} \setminus \{i, j\},$$

$$\hat{\delta}_i(t) = 0, \quad \hat{\delta}_j(t) = \delta_i^*(t) + \delta_j^*(t),$$

$$\hat{\delta}_k(t) = \delta_k^*(t), \quad k \in \mathcal{N} \setminus \{i, j\},$$

$$\hat{\theta}_i(t) = \theta_i^*(t) + \frac{1}{y_{ij}}(\gamma_i^*(t) - \delta_i^*(t)),$$

$$\hat{\theta}_k(t) = \theta_k^*(t), k \in \mathcal{N} \setminus \{i\},$$

$$\hat{b}_i = 0, \quad \hat{b}_j = b_i^* + b_j^*,$$

$$\hat{b}_k = b_k^*, \quad k \in \mathcal{N} \setminus \{i, j\},$$

$$\hat{p}_{ij}(t) = p_{ij}^*(t) + \gamma_i^*(t) - \delta_i^*(t),$$

$$\hat{p}_{kl}(t) = p_{kl}^*(t), \quad k \sim l, (k, l) \neq (i, j).$$

We do this successively for each $i \in \mathcal{K}$ to obtain a feasible point of $\Pi^{\mathcal{K}}$. Since the generation

profiles remained invariant, the resulting point is optimal for $\Pi^{\mathcal{K}}$. This completes the proof of Theorem 1.

B. Discussion

First, we explore a few practical power networks, where Theorem 1 applies, i.e., network topologies with generator buses that have single connections. In particular, consider the networks shown in Figure 3. The single generator single load case in Figure 3a models topologies where generators and loads are geographically separated and are connected by a transmission line, e.g., see [45]. This is common where the inputs for the generation technology (like coal or natural gas) are available far away from where the loads are located in a network. Figure 3b is an example of a radial network, i.e., an acyclic graph. Most distribution networks conform to this topology, e.g., see [38], [46]. Also, isolated transmission networks, e.g., the power network in Catalina island [13] are radial in nature. Theorem 1 also applies to generic mesh networks that have generator buses with single connections, e.g., the network in Figure 2.

In all these examples, Theorem 1 implies that for any available storage budget h , it is always optimal to place *no storage* on generator buses that have single connections. Problem P , in general, has multiple optimal solutions, but Theorem 1 proves the existence of an optimum with the property that $b_i^* = 0$ for all i in \mathcal{K} .

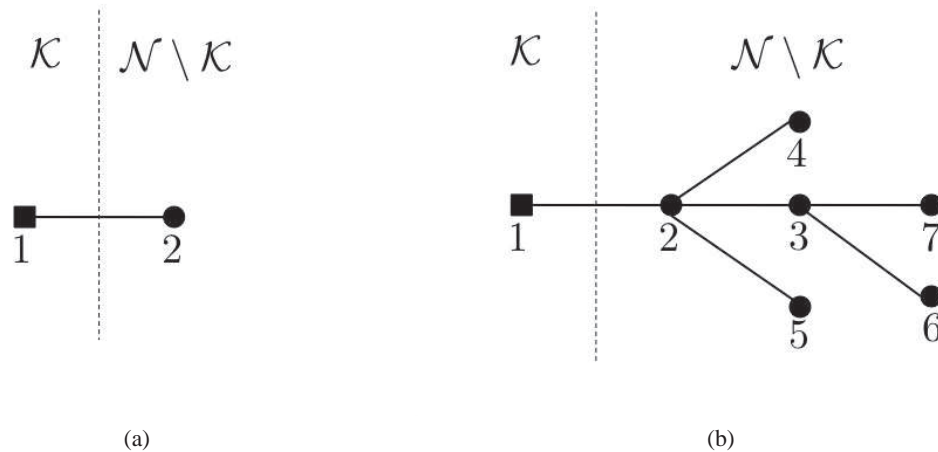


Fig. 3: Examples of power networks (a) Single generator single load system (b) A radial network.

Our result is robust to changes in demand profiles, generation capacities, line flow capacities

and admittances in the entire network, i.e., it remains optimal not to place any storage at buses in set \mathcal{K} under any changes to the above-mentioned parameters. The optimal sizing and the operation of the storage devices, however, might vary with changes in these parameters. To illustrate the efficacy of this robustness, consider the example in Figure 3a. Suppose the line flow capacity is larger than the peak value of the demand profile, i.e., $f_{12} \geq \max_{t \in [T]} d_2(t)$. It can be checked that placing all the available storage at the generator bus is an optimal solution. If at a later time during the operation of the network, the demand increases such that the peak demand surpasses the line capacity, this placement of storage no longer remains optimal and requires new infrastructure for storage to be built on the demand side to avoid load shedding. If, however, we use the optimum as suggested by the problem $\Pi^{\mathcal{K}}$ and place all storage on the demand side from the beginning, then this placement not only can accommodate the change in the demand, but, it also, remains optimal under the available storage budget. To explore a different direction, suppose another generator is built to supply the load in Figure 3a. Then, Theorem 1 implies that the optimal placement still has no storage at bus 1 and thus is robust to such extensions of the network. In summary, Theorem 1 suggests that the optimal storage placement problem has an interesting underlying structure and the solution to this problem is robust to changes in a wide class of system parameters and hence useful for network planners.

The network planner uses Theorem 1 to solve the investment decision strategy and the optimal control policy for the generation and storage devices for a power network as follows. Let \mathcal{K} be the set of *all* generator buses with single connections. With the demand profiles and network parameters as input, solve the problem $\Pi^{\mathcal{K}}$ optimally. The optimal storage capacities b_k^* , $k \in \mathcal{N} \setminus \mathcal{K}$ define the investment decision strategy for sizing storage units at different buses. The optimal generation profiles $g_k^*(t)$, $k \in \mathcal{N}_G$ define the economic dispatch of the various generators and the optimal charging/ discharging profiles $\gamma_k^*(t)$, $\delta_k^*(t)$, $k \in \mathcal{N} \setminus \mathcal{K}$ define the optimal control of the installed storage units.

We end this discussion with a simple remark regarding the incorporation of renewables to the power network. Renewable generators with marginal cost of production can be treated as negative loads (and the corresponding buses as load buses). The result of Theorem 1 still applies to all conventional generator buses that have single links. However, buses with single connections with a renewable energy source differ from the corresponding ones with conventional sources and thus are not guaranteed to have zero storage in the optimal placement.

C. On generators with multiple connections

Generator buses with multiple connections may not always have zero storage capacity in the optimal allocation. In this section, we illustrate this fact through a simple example. Consider a 3-node network as shown in Figure 4. All quantities are in per units. Let the cost of generation at node 1 be $c_1(g_1) = g_1^2$. Let $T = 4$ and the demand profiles at nodes 2 and 3 be

$$d_2 = (9, 10, 0, 10) \quad \text{and} \quad d_3 = (0, 10, 10, 10).$$

Also, suppose that the line capacities are $f_{12} = f_{13} = 9.5$ and the available storage budget is $h = 5$. Finally, assume no losses and ignore the ramp constraints in the charging and discharging processes, i.e. $\alpha = 1$ and $\epsilon_\gamma = \epsilon_\delta = 1$. It can be checked that $p_* = 877 < \pi_*^{\{1\}} = 900.75$.

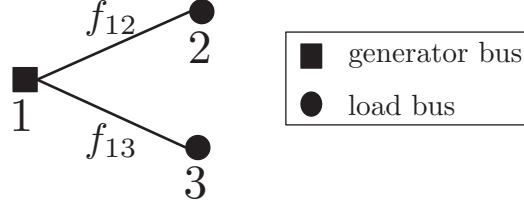


Fig. 4: A network with a generator that has multiple connections.

IV. RESULTS ON SPECIFIC NETWORK TOPOLOGIES

In problems P and $\Pi^{\mathcal{K}}$, we solve for the optimal placement and control of storage in a power-network, given the demand profiles $d_k(t), t \in [T]$, the storage budget h , the capacities of the generators $\bar{g}_k, k \in \mathcal{N}_G$ and other network parameters such as the line flow limits $f_{kl}, k \sim l$. In this section, we explore the behavior of the optimal cost of production as a function of these parameters. This provides valuable insights on various design issues, e.g., how much savings in terms of generation cost do we achieve by investing in an extra unit of storage. We explore such questions for specific network topologies.

We make a few simplifying assumptions in this section. Let $c_k(\cdot), k \in \mathcal{N}_G$ be strictly convex and let $\alpha = 1$ and $\epsilon_\gamma = \epsilon_\delta = 1$. The proofs of the results are included in the Appendix.

A. Single Generator Single Load Network

Consider the single generator single load network shown in Figure 5. Generator at bus 1 is connected to a load (or demand) at bus 2 using a single line, i.e., $\mathcal{K} = \mathcal{N}_G = \{1\}$ and $\mathcal{N}_D = \{2\}$.

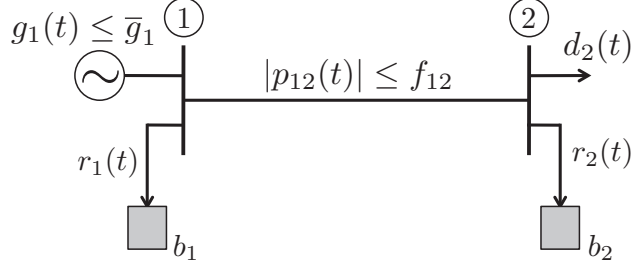


Fig. 5: Single generator single load network. Available storage budget is $h \geq b_1 + b_2$.

For this network, placing all the available storage resources at the load bus is always optimal. This is an immediate consequence of Theorem 1. In this section, for any fixed demand profile $d_2(t), t \in [T]$ of the load bus, we analyze the behavior of the optimal cost of production as a function of the generation capacity \bar{g}_1 , the line flow capacity f_{12} and the available storage budget h ; in particular, let the parameterized storage placement problem be $P(\bar{g}_1, f_{12}, h)$ and its optimal cost be $p_*(\bar{g}_1, f_{12}, h)$. Similarly define, $\Pi^{\{1\}}(\bar{g}_1, f_{12}, h)$ and $\pi_*^{\{1\}}(\bar{g}_1, f_{12}, h)$.

At the optimum of $P(\bar{g}_1, f_{12}, h)$, we have $g_1^*(t) \leq f_{12}, t \in [T]$ from Lemma 2. Also, it satisfies $g_1^*(t) \leq \bar{g}_1, t \in [T]$. Thus, to characterize the optimal point of $P(\bar{g}_1, f_{12}, h)$, it is equivalent to consider the constraint $g_1(t) \leq \min \{\bar{g}_1, f_{12}\}, t \in [T]$.

Proposition 1. For any $h \geq 0$, problem $P(\bar{g}_1, f_{12}, h)$ is feasible iff $\min \{\bar{g}_1, f_{12}\} \geq f_{\min}$, where

$$f_{\min} = \max \left\{ \max_{1 \leq t \leq T} \left(\frac{\sum_{\tau=1}^t d_2(\tau)}{t} \right), \max_{1 \leq t_1 < t_2 \leq T} \left(\frac{\sum_{\tau=t_1+1}^{t_2} d_2(\tau) - h}{t_2 - t_1} \right) \right\}. \quad (16)$$

Moreover, if $\min \{\bar{g}_1, f_{12}\} \geq f_{\min}$, then $p_*(\bar{g}_1, f_{12}, h) = p_*(f_{\min}, f_{\min}, h)$.

We interpret this result as follows. If either the line flow limit $f_{12} < f_{\min}$ or the generation capacity $\bar{g}_1 < f_{\min}$, the load cannot be satisfied. Notice that f_{\min} for $h > 0$ is no more than f_{\min} for $h = 0$. Thus, storage can be used to reduce the cost of operation avoiding transmission upgrades and generation capacity expansion [30]. Interestingly, for $f_{12} \geq f_{\min}$ and $\bar{g}_1 \geq f_{\min}$, the optimal cost of operation does not depend on the specific values of f_{12} and \bar{g}_1 . From transmission or distribution planning perspective, investment in line and generation capacities over f_{\min} do not reduce the cost of operation. We provide an illustrative example at the end of this section.

Next, we characterize the behavior of $P(\bar{g}_1, f_{12}, h)$ and its optimal cost $p^*(\bar{g}_1, f_{12}, h)$ as a func-

tion of h . For a given f_{12} and \bar{g}_1 , the minimum required storage budget to serve the load depends on the demand profile $d_2(t), t \in [T]$. This may or may not be zero, depending on $d_2(t), t \in [T]$, f_{12} and \bar{g}_1 . We calculate this minimum required storage budget, (say h_{min}) in Proposition 2. Also, it is easy to observe that as we allow larger storage budget, the generation cost does not reduce beyond a point, i.e., there exists h_{sat} such that $p_*(\bar{g}_1, f_{12}, h) = p_*(\bar{g}_1, f_{12}, h_{sat})$ for all $h \geq h_{sat}$. We also calculate h_{sat} in Proposition 2. First, we introduce some notation. Construct the sequence $\{\tau_m\}_{m=0}^M$ as follows. Let $\tau_0 = 0$. Define τ_m iteratively:

$$\tau_m = \arg \max_{\tau_{m-1}+1 \leq t \leq T} \left(\frac{\sum_{\tau=\tau_{m-1}+1}^t d_2(\tau)}{t - \tau_{m-1}} \right), \quad (17)$$

for $1 \leq m \leq M$, where M is the smallest integer for which $\tau_M = T$. Note that the sequence depends only on the demand profile $d_2(t), t \in [T]$. For any $x \in \mathbb{R}$, let $[x]^+ := \max(x, 0)$.

Proposition 2. *Problem $P(\bar{g}_1, f_{12}, h)$ satisfies:*

- (a) *If $\min \{\bar{g}_1, f_{12}\} < \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d_2(\tau)}{t} \right)$, then $P(\bar{g}_1, f_{12}, h)$ is infeasible for all $h \geq 0$.*
- (b) *Suppose, $\min \{\bar{g}_1, f_{12}\} \geq \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d_2(\tau)}{t} \right)$. Then, $P(\bar{g}_1, f_{12}, h)$ is feasible iff $h \geq h_{min}$ and $p_*(\bar{g}_1, f_{12}, h)$ is convex and non-increasing in h , where*

$$h_{min} = \max_{0 \leq t_1 \leq t_2 \leq T} \left[\sum_{\tau=t_1+1}^{t_2} (d_2(\tau) - \min \{\bar{g}_1, f_{12}\}) \right]^+. \quad (18)$$

Furthermore, $p_*(\bar{g}_1, f_{12}, h)$ is constant for all $h \geq h_{sat}$, where

$$h_{sat} = \max_{1 \leq m \leq M} \left[\max_{\tau_{m-1}+1 \leq t \leq \tau_m} \left\{ \left(\sum_{\tau=\tau_{m-1}+1}^{\tau_m} d_2(\tau) \right) \frac{t - \tau_{m-1}}{\tau_m - \tau_{m-1}} - \left(\sum_{\tau=\tau_{m-1}+1}^t d_2(\tau) \right) \right\} \right]. \quad (19)$$

The condition $\min \{\bar{g}_1, f_{12}\} \geq \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d_2(\tau)}{t} \right)$ implies that there is some $h > 0$ for which $P(\bar{g}_1, f_{12}, h)$ is feasible. If this condition is violated, the problem remains infeasible no matter how large the storage budget h is. More the storage budget, lesser is the generation cost and hence $p^*(\bar{g}_1, f_{12}, h)$ is decreasing in h . The convexity, however, implies that there is diminishing marginal returns on the investment on storage, i.e., the benefit of the first unit installed is more than that from the second unit. As a final note, observe that h_{sat} is a function of only the demand profile and is independent of the generation and line flow capacities.

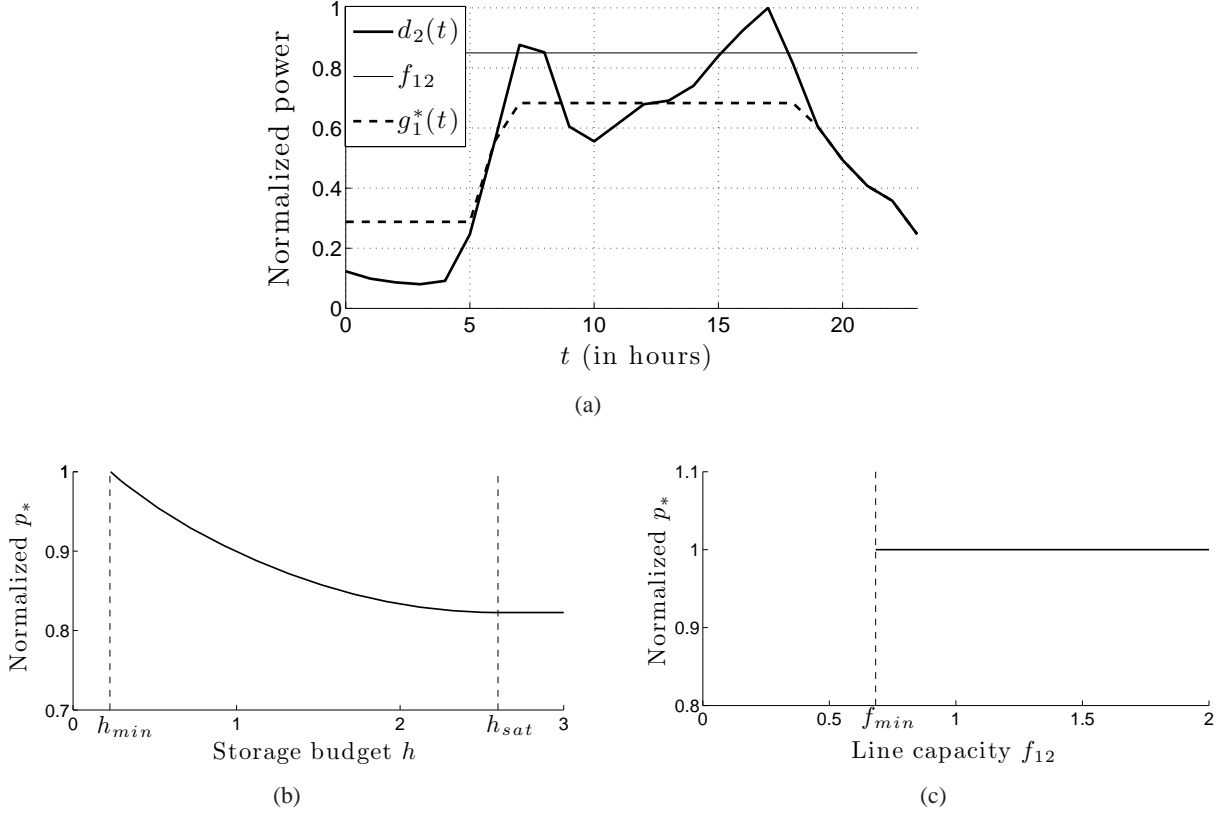


Fig. 6: Plots to illustrate propositions 1 and 2. (a) Typical hourly load profile and optimal generation portfolio for line flow capacity $f_{12} = 0.85$, generation capacity $\bar{g}_1 = 1$ and storage budget $h = 1$ (b) $p_*(\bar{g}_1 = 1, f_{12} = 0.85, h)$. (c) $p_*(\bar{g}_1 = 1, f_{12}, h = 1)$.

Example: Now we explain propositions 1 and 2 with an example. All quantities are in per units. Consider an hourly load profile $d_2(t), t \in [T]$ as shown in Figure 6a. The optimal generation profile $g_1^*(t), t \in [T]$ for $P(\bar{g}_1 = 1, f_{12} = 0.85, h = 1)$ has been plotted in the same Figure. Notice that $\max_{t \in [T]} g_1^*(t) \leq f_{12}$ as stated in Lemma 2.

Consider the plots in Figures 6b and 6c. We plot $p^*(\bar{g}_1 = 1, f = 0.85, h)$ for h in $[0, 3]$ in Figure 6b. Notice that $f_{12} \leq \max_{t \in [T]} d_2(t)$, i.e., the problem is infeasible in the absence of storage. We calculate $h_{min} = 0.226$ and $h_{sat} = 2.598$ from proposition 2. In Figure 6c, we plot $p^*(\bar{g}_1 = 1, f_{12}, h = 1)$ for f_{12} in $[0, 2]$. As in proposition 1, the problem is infeasible for $f_{12} < f_{min} = 0.683$ and the optimal cost remains constant for $f_{12} \geq f_{min}$.

B. Star Network

Consider a star network on $n \geq 2$ nodes as shown in Figure 7. $\mathcal{N}_G = \{1\}$ and $\mathcal{N}_D = \{2, 3, \dots, n\}$. For fixed demand profiles $d_k(t), t \in [T], k \in \mathcal{N}_D$, line flow capacities $f_{1k}, k \in \mathcal{N}_D$ and capacity of the generator \bar{g}_1 , let $P(h)$ and $\Pi^{\{1\}}(h)$ denote the storage placement problem and its restricted version as functions of the available storage budget h . Also, let $p_*(h)$ and $\pi_*^{\{1\}}(h)$ be their optimal costs respectively.

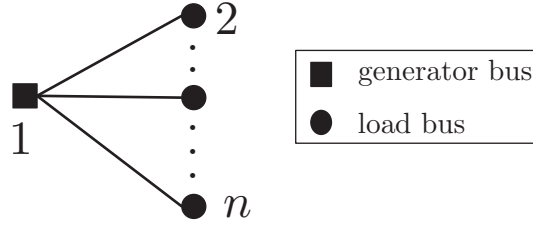


Fig. 7: A star network on n buses.

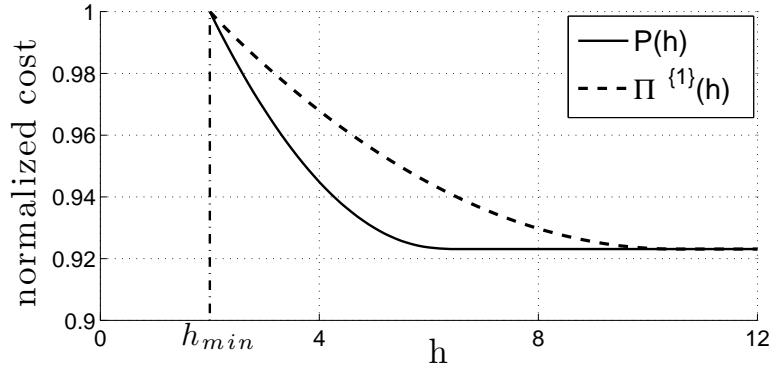


Fig. 8: $P(h)$ and $\Pi^{\{1\}}(h)$ for the simple 3-node star network in Figure 4.

In Section III-C we showed that placing zero storage at the generator bus of a star network is not optimal, i.e., in general, $p_*(h) \neq \pi_*(h)$. In Figure 8, we plot $p_*(h)$ and $\pi_*(h)$ for the 3-node star network shown in Figure 4 over a range of values of the total storage budget h . Observe that $p_*(h) < \pi_*^{\{1\}}(h)$ for some values of h but they coincide at:

- Minimum value of h for which $P(h)$ and $\Pi^{\{1\}}(h)$ are feasible.
- Large enough values of h .

We formally state this for a general n -node star network in the following. Assume $\bar{g} = \infty$.

Proposition 3. Suppose $f_{1k} \geq \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d_k(\tau)}{t} \right)$ for all $k \in \mathcal{N}_D$. Then, $P(h)$ and $\Pi^{\{1\}}(h)$ are feasible iff $h \geq h_{min}$, where

$$h_{min} = \sum_{k \in \mathcal{N}_D} \max_{0 \leq t_1 < t_2 \leq T} \left[\sum_{\tau=t_1+1}^{t_2} (d_k(\tau) - f_{1k}) \right]^+. \quad (20)$$

Moreover:

- (a) $p_*(h_{min}) = \pi_*^{\{1\}}(h_{min})$,
- (b) There exists $h_o \geq h_{min}$ such that $p_*(h) = \pi_*^{\{1\}}(h)$ for all $h \geq h_o$.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we have derived analytic results on the optimal storage placement and sizing of storage units in the power grid. This provides valuable information for transmission and distribution system planners to find a suitable investment strategy in large-scale storage for the future smart grid. This work suggests that the storage placement problem has an interesting underlying structure and can be exploited to make sound planning decisions for sizing storage, that are robust to changes in a wide class of network parameters. As a result, it provides a framework to find the optimal storage capacity allocations in a network and in addition finds the optimal control policy for the generators and storage devices.

A natural direction for future work is to study the same problem with performance metrics other than the production cost such as how storage can be used to defer transmission line upgrades or generation capacity expansion. This is a first step to “assess the technical and economic attributes” [28] of this emerging smart grid technology. Another interesting direction is characterizing the optimal allocation of storage with stochastic demand and generation. In particular, we believe that storage placement on generators with multiple connections with realistic stochastic demand models would provide us with valuable insights.

VI. ACKNOWLEDGEMENTS

The authors gratefully acknowledge Prof. K. Mani Chandy at California Institute of Technology and Mr. Paul DeMartini from Resnick Institute for their helpful comments.

REFERENCES

- [1] R. Schainker, "Executive overview: Energy storage options for a sustainable energy future," in *Proc. of IEEE PES General Meeting*, 2004, pp. 2309–2314.
- [2] A. Nourai, "Installation of the first dist. energy storage system at american electric power (aep)," *Sandia Nat. Labs*, 2007.
- [3] P. Denholm, E. Ela, B. Kirby, and M. Milligan, "The role of energy storage with renewable electricity generation," 2010.
- [4] P. Varaiya, F. Wu, and J. Bialek, "Smart operation of smart grid: Risk-limiting dispatch," *Proc. of the IEEE*, vol. 99, no. 1, pp. 40–57, 2011.
- [5] J. Eyer and G. Corey, "Energy storage for the elec. grid: Benefits and market potential assessment guide," *Sandia Nat. Lab.*, 2010.
- [6] J. Greenberger. (2011, nov) The smart grid's problem may be storage's opportunity. [Online]. Available: <http://theenergycollective.com/jim-greenberger/70813/smart-grids-problem-may-be-storages-opportunity>
- [7] D. Lindley, "Smart grids: The energy storage problem." *Nature*, vol. 463, no. 7277, p. 18, 2010.
- [8] "Dist. energy storage serving national interests: Advancing wide-scale des in the united states," *KEMA Inc*, April 2012.
- [9] S. Chu and A. Majumdar, "Opportunities and challenges for a sustainable energy future," *Nature*, vol. 488, no. 7411, pp. 294–303, 2012.
- [10] D. Rastler, *Electricity Energy Storage Technology Options: A White Paper Primer on Applications, Costs and Benefits*. Electric Power Research Institute, 2010.
- [11] "2020 strategic analysis of energy storage in california," *California Energy Commision*, November 2011.
- [12] C. de Morsella. (2011) Fifteen grid scale energy storage solutions to watch. [Online]. Available: <http://greeneconomypost.com/fifteen-grid-scale-energy-storage-solutions-watch-15924.htm>
- [13] H. Xu, U. Topcu, S. Low, C. Clarke, and K. Chandy, "Load-shedding probabilities with hybrid renewable power generation and energy storage," in *Communication, Control, and Computing (Allerton), 2010 48th Annual Allerton Conference on*. IEEE, 2010, pp. 233–239.
- [14] D. Biello. Storing the breeze: New battery might make wind power more reliable. [Online]. Available: <http://www.scientificamerican.com/article.cfm?id=storing-the-breeze-new-battery-might-make-wind-power-reliable>
- [15] C. Budischak, D. Sewell, H. Thomson, L. Mach, D. Veron, and W. Kempton, "Cost-minimized comb. of wind power, solar power and electrochemical storage, powering the grid up to 99.9% of the time," *Journal of Power Sources*, 2012.
- [16] M. Campbell. Apple's wind turbine technology uses heat, not rotational energy to generate electricity. [Online]. Available: <http://appleinsider.com/articles/12/12/27/apples-wind-turbine-technology-uses-heat-not-kinetic-energy-to-generate-electricity>
- [17] "Electrical energy storage," *IEC WHITE PAPER EES ed1.0*, 2011.
- [18] H. Oh, "Optimal planning to include storage devices in power systems," *Pow. Sys., IEEE Trans. on*, vol. 26, no. 3, pp. 1118–1128, 2011.
- [19] Y. M. Atwa and E. F. El-Saadany, "Optimal allocation of ESS in distribution systems with a high penetration of wind energy," *IEEE Trans. on Power Sys.*, vol. 25, no. 4, pp. 1815–1822, Nov. 2010.
- [20] H. Su and A. Gamal, "Modeling and analysis of the role of fast-response energy storage in the smart grid," *arXiv preprint arXiv:1109.3841*, 2011.
- [21] Y. Kanoria, A. Montanari, D. Tse, and B. Zhang, "Distributed storage for intermittent energy sources: Control design and performance limits," in *Communication, Control, and Computing (Allerton), 2011 49th Annual Allerton Conference on*. IEEE, 2011, pp. 1310–1317.

- [22] I. Koutsopoulos, V. Hatzi, and L. Tassiulas, "Optimal energy storage control policies for the smart power grid," in *Smart Grid Communications (SmartGridComm), 2011 IEEE International Conference on*. IEEE, 2011, pp. 475–480.
- [23] D. Gayme and U. Topcu, "Optimal power flow with large-scale energy storage integration," *IEEE Trans. on Power Sys.*, to appear, 2012.
- [24] M. Chandy, S. Low, U. Topcu, and H. Xu, "A simple optimal power flow model with energy storage," in *Proc. of Conf. on Decision and Ctrl.*, 2010.
- [25] A. R. Bergen and V. Vittal, *Power Systems Analysis*, 2nd ed. Prentice Hall, 2000.
- [26] K. Purchala, L. Meeus, D. Van Dommelen, and R. Belmans, "Usefulness of dc power flow for active power flow analysis," in *Power Engineering Society General Meeting, 2005. IEEE*. IEEE, 2005, pp. 454–459.
- [27] K. S. Pandya and S. K. Joshi, "A survey of optimal power flow methods," *J. of Theoretical and App. Info. Tech.*, vol. 4, no. 5, pp. 450–458, 2008.
- [28] M. Hoffman and A. Sadovsky, "Analysis tools for sizing and placement of energy storage in grid app., a lit. rev." 2010.
- [29] M. Kraning, Y. Wang, E. Akuiyibo, and S. Boyd, "Operation and configuration of a storage portfolio via convex optimization," in *Proc. of the IFAC World Congress*, 2010, pp. 10 487–10 492.
- [30] P. Denholm and R. Sioshansi, "The value of compressed air energy storage with wind in transmission-constrained electric power systems," *Energy Policy*, vol. 37, pp. 3149–3158, 2009.
- [31] P. Harsha and M. Dahleh, "Optimal management and sizing of energy storage under dynamic pricing for the efficient integration of renewable energy," in preparation.
- [32] Univ. of Washington, "Power systems test case archive." [Online]. Available: <http://www.ee.washington.edu/research/pstca/>
- [33] A. E. Sjödin, D. Gayme, and U. Topcu, "Risk-mitigated optimal power flow for wind powered grids," in *Proc. of the American Ctrl. Conf.*, 2012.
- [34] S. Bose, F. Gayme, U. Topcu, and K. Chandy, "Optimal placement of energy storage in the grid," in *Decision and Control (CDC), 2012 51st IEEE Conference on*.
- [35] X. Bai, H. Wei, K. Fujisawa, and Y. Wang, "Semidefinite programming for optimal power flow problems," *Int'l J. of Electrical Power & Energy Sys.*, vol. 30, no. 6-7, pp. 383–392, 2008.
- [36] X. Bai and H. Wei, "Semi-definite programming-based method for security-constrained unit commitment with operational and optimal power flow constraints," *IET Generation, Transmission & Distribution*, vol. 3, no. 2, pp. 182–197, 2009.
- [37] J. Lavaei and S. Low, "Zero duality gap in optimal power flow problem," *IEEE Trans. on Power Sys.*, vol. 27, Feb. 2012.
- [38] S. Bose, D. Gayme, S. Low, and K. Chandy, "Quadratically constrained quadratic programs on acyclic graphs with application to power flow," *To appear in IEEE Transactions on Automatic Control*, 2012.
- [39] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge Univ. Press, 2004.
- [40] H. Wolkowicz, R. Saigal, and L. Vandenberghe, *Handbook of semidefinite programming: theory, algorithms, and applications*. Springer Netherlands, 2000, vol. 27.
- [41] C. Thampoulidis, S. Bose, and B. Hassibi, "Optimal large-scale storage placement in single generator single load networks," 2012, accepted to IEEE PES General Meeting 2013.
- [42] —, "On the distribution of energy storage in electricity grids," 2013, submitted to Decision and Control (CDC), 2013 52nd IEEE Conference on.
- [43] Hourly load data. [Online]. Available: <http://www.pjm.com/markets-and-operations/energy/real-time/loadhryr.aspx>
- [44] M. Korpaas, A. T. Holen, and R. Hildrum, "Operation and sizing of energy storage for wind power plants in a market system," *International Journal of Electrical Power & Energy Systems*, vol. 25, no. 8, pp. 599–606, 2003.

- [45] L. Gaillac, J. Castaneda, A. Edris, D. Elizondo, C. Wilkins, C. Vartanian, and D. Mendelsohn, “Tehachapi wind energy storage project: Description of operational uses, system components, and testing plans,” in *Transmission and Distribution Conference and Exposition (T D), 2012 IEEE PES*, may 2012, pp. 1–6.
- [46] IEEE Distribution Test Feeders. [Online]. Available: <http://www.ewh.ieee.org/soc/pes/dsacom/testfeeders/index.html>

APPENDIX

Here, we present the proofs of the results presented in Section IV. For the single generator single node and the star network, we drop the voltage angles $\theta_k(t), k \in \mathcal{N}, t \in [T]$. For any value of the power flow $p_{1k}(t)$ from bus 1 to bus k , voltage angles $\theta_k(t)$ can always be chosen to satisfy the power flow constraint in (2).

Furthermore, since $\alpha = 1$, define $r_k(t) := \gamma_k(t) - \delta_k(t)$ be the power that flows into the storage device at node $k \in \mathcal{N}$ at time $t \in [T]$. Notice that $r_k(t)$ can be positive or negative depending on whether power flows in or out of the storage device. Also, the storage level of the storage device at node $k \in \mathcal{N}$, at time t can be written as $s_k(t) = \sum_{\tau=1}^t r_k(\tau)$.

A. Proofs for Single Generator Single Load Networks

We drop subscripts from the variables $d_2(t), g_1(t), t \in [T]$, $f_{12}, \bar{g}_1, c_1(\cdot)$ and the superscript from $\Pi^{\{1\}}(\cdot), \pi_*^{\{1\}}(\cdot)$ for ease of notation throughout this section.

Proposition 1. *For any $h \geq 0$, problem $P(\bar{g}, f, h)$ is feasible iff $\min \{\bar{g}, f\} \geq f_{\min}$, where*

$$f_{\min} = \max \left\{ \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d(\tau)}{t} \right), \max_{1 \leq t_1 < t_2 \leq T} \left(\frac{\sum_{\tau=t_1+1}^{t_2} d(\tau) - h}{t_2 - t_1} \right) \right\}. \quad (21)$$

Moreover, if $\min \{\bar{g}, f\} \geq f_{\min}$, then $p_*(\bar{g}, f, h) = p_*(f_{\min}, f_{\min}, h)$.

Proof: From Theorem 1, it suffices to show the claim for $\Pi(\bar{g}, f, h)$ and $\pi_*(\bar{g}, f, h)$. First, we show that if $\Pi(\bar{g}, f, h)$ is feasible, then $\min \{\bar{g}, f\} \geq f_{\min}$. Fix any $h \geq 0$ and let $g(t), t \in [T]$ be a feasible generation profile. Since $\sum_{\tau=1}^t r_2(\tau) = s_2(t) \geq 0$, we have for any $t \in [T]$

$$\max_{t' \in [T]} g(t') \geq \frac{\sum_{\tau=1}^t g(\tau)}{t} = \frac{\sum_{\tau=1}^t d(\tau) + \sum_{\tau=1}^t r_2(\tau)}{t} \geq \frac{\sum_{\tau=1}^t d(\tau)}{t}. \quad (22)$$

Furthermore, for any $1 \leq t_1 < t_2 \leq T$, the power extracted from the storage device between

time instants t_1 and t_2 cannot exceed the total storage budget h and hence we have

$$\max_{t' \in [T]} g(t') \geq \frac{\sum_{\tau=t_1+1}^{t_2} g(\tau)}{t_2 - t_1} = \frac{\sum_{\tau=t_1+1}^{t_2} d(\tau) + \sum_{\tau=t_1+1}^{t_2} r_2(\tau)}{t_2 - t_1} \geq \frac{\sum_{\tau=t_1+1}^{t_2} d(\tau) - h}{t_2 - t_1}. \quad (23)$$

Since $g(t), t \in [T]$ is feasible, $g(t) \leq \min \{\bar{g}, f\}$ for all $t \in [T]$. Hence, combining (22) and (23), we get

$$\min \{\bar{g}, f\} \geq \max_{t' \in [T]} g(t') \geq f_{\min}.$$

Next, we show that $\min \{\bar{g}, f\} \geq f_{\min}$ is sufficient for $\Pi(\bar{g}, f, h)$ to be feasible. Consider the optimal generation profile $g^*(t), t \in [T]$ for the relaxed problem $\Pi(+\infty, +\infty, h)$. Suppose it satisfies

$$\max_{t \in [T]} g^*(t) \leq f_{\min}. \quad (24)$$

Then $g^*(t), t \in [T]$ is also feasible and optimal for problem $\Pi(\bar{g}, f, h)$ for $\min \{\bar{g}, f\} \geq f_{\min}$. Also, $\pi_*(\bar{g}, f, h) = \pi_*(f_{\min}, f_{\min}, h)$ for $\min \{\bar{g}, f\} \geq f_{\min}$. It remains to show that (24) indeed holds. Consider the following notation.

$$t_{\max} := \max\{t \in [T] \mid g^*(t) = \max_{\tau \in [T]} g^*(\tau)\},$$

$$t_{\text{less}} := \{0 \leq t < t_{\max} \mid g^*(t) < g^*(t_{\max})\}.$$

In the above definition $g^*(0) := 0$ for convenience. If $g^*(t_{\max}) = 0$, then (24) clearly holds. Henceforth, assume $g^*(t_{\max}) > 0$. Then, $g^*(t), t \in [T]$ satisfies:

$$\begin{aligned} \max_{t \in [T]} g^*(t) &= \frac{\sum_{\tau=t_{\text{less}}+1}^{t_{\max}} g^*(\tau)}{t_{\max} - t_{\text{less}}} \\ &= \frac{\sum_{\tau=t_{\text{less}}+1}^{t_{\max}} [d(\tau) + r_2^*(\tau)]}{t_{\max} - t_{\text{less}}} \\ &= \frac{1}{t_{\max} - t_{\text{less}}} \left[\left(\sum_{\tau=t_{\text{less}}+1}^{t_{\max}} d(\tau) \right) + s_2^*(t_{\max}) - s_2^*(t_{\text{less}}) \right] \end{aligned} \quad (25)$$

Now, suppose the following holds:

$$s_2^*(t_{\max}) = 0 \quad \text{and} \quad s_2^*(t_{\text{less}}) = \begin{cases} 0, & \text{if } t_{\text{less}} = 0, \\ h, & \text{otherwise.} \end{cases} \quad (26)$$

If (26) holds, it follows from (25):

$$\begin{aligned} \max_{t \in [T]} g^*(t) &= \begin{cases} \frac{\sum_{\tau=1}^{t_{max}} d(\tau)}{t_{max}}, & \text{if } t_{less} = 0, \\ \frac{\sum_{\tau=t_{less}+1}^{t_{max}} d(\tau) - h}{t_{max} - t_{less}}, & \text{otherwise,} \end{cases} \\ &\leq f_{\min}. \end{aligned}$$

and hence (24) is satisfied. Next, we show that (26) indeed holds to complete the proof. First we prove that $s_2^*(t_{max}) = 0$, i.e., the storage device at node 2 fully discharges at time t_{max} . Suppose $s_2^*(t_{max}) > 0$. As in Lemma 2, we construct a modified generation profile and storage control policy that is feasible and has an objective function value no greater than $\pi_*(+\infty, +\infty, h)$. But, the optimal generation profile $g^*(t), t \in [T]$ is unique since the cost function $c(\cdot)$ is assumed to be strictly convex. Hence we derive a contradiction. By hypothesis, $s_2^*(t_{max}) > 0$ and hence the storage device at bus 2 discharges for some $t > t_{max}$. Let t_1 be the first such time instant and define

$$\Delta_1 := \min \{s_2^*(t_{max}), g^*(t_{max}), g^*(t_{max}) - g^*(t_1)\}.$$

Notice that $\Delta_1 > 0$. Consider the modified generation profile $\tilde{g}(t)$ and control policy $\tilde{r}_2(t)$, that differ from $g^*(t)$ and $r_2^*(t)$ only at t_{max} and t_1 as follows:

$$\begin{aligned} \tilde{g}(t_{max}) &= g^*(t_{max}) - \Delta_1, & \tilde{g}(t_1) &= g^*(t_1) + \Delta_1, \\ \tilde{r}_2(t_{max}) &= r_2^*(t_{max}) - \Delta_1, & \tilde{r}_2(t_1) &= r_2^*(t_1) + \Delta_1. \end{aligned}$$

Using Lemma 1, we have

$$c(\tilde{g}(t_{max})) + c(\tilde{g}(t_1)) \leq c(g^*(t_{max})) + c(g^*(t_1)).$$

It can be checked that the modified profiles are feasible for $\Pi(+\infty, +\infty, h)$. The details are omitted for brevity. This is a contradiction and hence $s_2^*(t_{max}) = 0$.

Next, we characterize $s_2^*(t_{less})$. If $t_{less} = 0$, then $s_2^*(t_{less}) = s_2^0 = 0$. If $t_{less} > 0$, we prove that $s_2^*(t_{less}) = h$, i.e., the storage device at node 2 is fully charged at time t_{less} . Suppose $s_2^*(t_{less}) < h$. As above, we construct a modified generation profile $\tilde{g}(t)$ and storage control policy $\tilde{r}_2(t)$ that achieves an objective value no greater than $\pi_*(+\infty, +\infty, h)$ to derive a contradiction.

In particular, define

$$\Delta_2 := \min \{h - s_2^*(t_{less}), g^*(t_{less} + 1), g^*(t_{less} + 1) - g^*(t_{less})\} > 0.$$

Consider $\tilde{g}(t)$ and $\tilde{r}_2(t)$, that differ from $g^*(t)$ and $r_2^*(t)$ only at t_{less} and $t_{less} + 1$ as follows:

$$\begin{aligned}\tilde{g}(t_{less}) &= g^*(t_{less}) + \Delta_2, & \tilde{g}(t_{less} + 1) &= g^*(t_{less} + 1) - \Delta_2, \\ \tilde{r}_2(t_{less}) &= r_2^*(t_{less}) + \Delta_2, & \tilde{r}_2(t_{less} + 1) &= r_2^*(t_{less} + 1) - \Delta_2.\end{aligned}$$

As above, this defines a feasible point for $\Pi(+\infty, +\infty, h)$ and achieves an objective value strictly less than $\pi_*(+\infty, +\infty, h)$. This is a contradiction and hence $s_2^*(t_{less}) = h$ for $t_{less} > 0$. ■

Proposition 2. *Problem $P(\bar{g}, f, h)$ satisfies:*

- (a) *If $\min \{\bar{g}, f\} < \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d(\tau)}{t} \right)$, then $P(\bar{g}, f, h)$ is infeasible for all $h \geq 0$.*
- (b) *Suppose, $\min \{\bar{g}, f\} \geq \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d(\tau)}{t} \right)$. Then, $P(\bar{g}, f, h)$ is feasible iff $h \geq h_{\min}$ and $p_*(\bar{g}, f, h)$ is convex and non-increasing in h , where*

$$h_{\min} = \max_{0 \leq t_1 \leq t_2 \leq T} \left[\sum_{\tau=t_1+1}^{t_2} (d(\tau) - \min \{\bar{g}, f\}) \right]^+. \quad (27)$$

Furthermore, $p_*(\bar{g}, f, h)$ is constant for all $h \geq h_{\text{sat}}$, where

$$h_{\text{sat}} = \max_{1 \leq m \leq M} \left[\max_{\tau_{m-1}+1 \leq t \leq \tau_m} \left\{ \left(\sum_{\tau=\tau_{m-1}+1}^{\tau_m} d(\tau) \right) \frac{t - \tau_{m-1}}{\tau_m - \tau_{m-1}} - \left(\sum_{\tau=\tau_{m-1}+1}^t d(\tau) \right) \right\} \right]. \quad (28)$$

Proof: From Theorem 1, it suffices to prove the claim for $\Pi(\bar{g}, f, h)$ and $\pi_*(\bar{g}, f, h)$.

- (a) To the contrary of the statement of the Proposition suppose that $\min \{\bar{g}, f\} < \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d(\tau)}{t} \right)$ and $\Pi(\bar{g}, f, h)$ is feasible for some $h \geq 0$. Then, it follows directly from Proposition 1 that $\min \{\bar{g}, f\} \geq f_{\min} \geq \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d(\tau)}{t} \right)$, contradicting our hypothesis.
- (b) First we show that if $\Pi(\bar{g}, f, h)$ is feasible then $h \geq h_{\min}$. Suppose $\Pi(\bar{g}, f, h)$ is feasible. Then, for all $0 \leq t_1 < t_2 \leq T$ Proposition 1 implies that $\min \{\bar{g}, f\} \geq f_{\min} \geq (\sum_{\tau=t_1+1}^{t_2} d(\tau) - h) / (t_2 - t_1)$. Rearranging this we get $h \geq \sum_{\tau=t_1+1}^{t_2} (d(\tau) - \min \{\bar{g}, f\})$.

Also, $h \geq 0$ and hence:

$$h \geq \max_{0 \leq t_1 < t_2 \leq T} \left[\sum_{\tau=t_1+1}^{t_2} (d(\tau) - \min\{\bar{g}, f\}) \right]^+ = h_{min}.$$

Now we show that $h \geq h_{min}$ is sufficient for $\Pi(\bar{g}, f, h)$ to be feasible. The relation $h \geq h_{min}$ can be equivalently written as follows:

$$\min\{\bar{g}, f\} \geq \frac{\sum_{\tau=t_1+1}^{t_2} d(\tau) - h}{t_2 - t_1}, \quad \text{for all } 0 \leq t_1 < t_2 \leq T. \quad (29)$$

Also, by hypothesis, we have

$$\min\{\bar{g}, f\} \geq \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d(\tau)}{t} \right). \quad (30)$$

Combining (29) and (30), we get $\min\{\bar{g}, f\} \geq f_{min}$. Then, Proposition 1 implies that $\Pi(\bar{g}, f, h)$ is feasible. Convexity and non-decreasing nature of $p_*(\bar{g}, f, h)$ as a function of h follows from linear parametric optimization theory [39].

Finally, we prove that $p_*(\bar{g}, f, h)$ is constant for all $h \geq h_{sat}$, where h_{sat} is as defined in (28). The proof idea here is as follows. We construct the optimal generation profile $g^*(t), t \in [T]$ for the problem $\Pi(+\infty, +\infty, +\infty)$ and show that it is feasible and hence optimal for the problem $\Pi(\bar{g}, f, +\infty)$ provided $\min\{\bar{g}, f\} \geq \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d(\tau)}{t} \right)$ holds. Problem $\Pi(+\infty, +\infty, +\infty)$ can be re-written as follows.

$$\begin{aligned} & \underset{g(t), t \in [T]}{\text{minimize}} && \sum_{t=1}^T c_1(g(t)) \\ & \text{subject to} && g(t) \geq 0, \quad \sum_{\tau=1}^t (g(\tau) - d(\tau)) \geq 0, \quad t \in [T], \end{aligned} \quad (31a)$$

$$\sum_{\tau=1}^T g(\tau) = \sum_{\tau=1}^T d(\tau). \quad (31b)$$

Let the Lagrange multipliers in equations (31a)–(31b) be $\lambda(t)$, $\ell(t)$, $t \in [T]$ and ν , respectively. It can be checked that the following primal-dual pair satisfies the Karush-Kuhn-Tucker conditions and hence is optimal for the convex program $\Pi(+\infty, +\infty, +\infty)$ and its

Lagrangian dual [39]. We omit the details for brevity.

$$\begin{aligned}
g^*(t) &= \frac{\sum_{\tau=\tau_{m-1}+1}^{\tau_m} d(\tau)}{\tau_m - \tau_{m-1}}, \quad t = \tau_{m-1} + 1, \dots, \tau_m \quad \text{and} \quad m = 1, 2, \dots, M, \\
\ell^*(t) &= \begin{cases} c'(g^*(\tau_m)) - c'(g^*(\tau_m + 1)), & \text{if } t = \tau_m, \quad m = 1, 2, \dots, M-1 \\ 0, & \text{otherwise} \end{cases}, \quad t \in [T], \\
\lambda^*(t) &= 0, \quad t \in [T], \quad \text{and} \quad \nu^* = -c'(g^*(T)).
\end{aligned}$$

The above profile $g^*(t), t \in [T]$ of $\Pi(+\infty, +\infty, +\infty)$ satisfies:

$$\max_{t \in [T]} g^*(t) = \max_{t \in [T]} \frac{\sum_{\tau=1}^t d(\tau)}{t} \leq \min \{\bar{g}, f\},$$

and hence is feasible and optimal for $\Pi(\bar{g}, f, +\infty)$. Note that $\sum_{\tau=\tau_{m-1}+1}^{\tau_m} (g^*(\tau) - d(\tau)) = 0$ for all $1 \leq m \leq M$. Thus, for $\tau_{m-1} < t \leq \tau_m$, we have

$$\begin{aligned}
s_2^*(t) &= \sum_{\tau=\tau_{m-1}}^t (g^*(\tau) - d(\tau)) \\
&= \frac{\sum_{\tau=\tau_{m-1}+1}^{\tau_m} d(\tau)}{\tau_m - \tau_{m-1}} (t - \tau_{m-1}) - \sum_{\tau=\tau_{m-1}+1}^t d(\tau).
\end{aligned}$$

Maximizing the above relation over all $t \in [T]$ we get $\max_{t \in [T]} s_2^*(t) = h_{sat}$. Therefore, $g^*(t), t \in [T]$ is feasible and optimal for $\Pi(\bar{g}, f, h)$ provided that $h \geq h_{sat}$. ■

B. Proofs for Star Networks

Proposition 3. Suppose $f_{1k} \geq \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d_k(\tau)}{t} \right)$ for all $k \in \mathcal{N}_D$. Then, $P(h)$ and $\Pi^{\{1\}}(h)$ are feasible iff $h \geq h_{min}$, where

$$h_{min} = \sum_{k \in \mathcal{N}_D} \max_{0 \leq t_1 < t_2 \leq T} \left[\sum_{\tau=t_1+1}^{t_2} (d_k(\tau) - f_{1k}) \right]^+. \quad (33)$$

Moreover:

- (a) $p_*(h_{min}) = \pi_*^{\{1\}}(h_{min})$,
- (b) There exists $h_o \geq h_{min}$ such that $p_*(h) = \pi_*^{\{1\}}(h)$ for all $h \geq h_o$.

Proof: First we show that $h \geq h_{min}$ is necessary for $P(h)$ to be feasible. Consider any feasible solution of $P(h)$. For any $k \in \mathcal{N}_D$ and $0 \leq t_1 < t_2 \leq T$, we have $\sum_{\tau=t_1+1}^{t_2} r_k(\tau) \geq -b_k$, since the power extracted from the storage device at node k cannot exceed the corresponding storage capacity b_k . Also, for any $k \in \mathcal{N}_D$ the power flow on the line joining buses 1 and k satisfies $p_{1k}(t) = d_k(t) + r_k(t) \leq f_{1k}$ for all $t \in [T]$. Combining the above relations and rearranging, we get $b_k \geq \sum_{\tau=t_1+1}^{t_2} (d_k(\tau) - f_{1k})$. Also for $k \in \mathcal{N}_D$, $b_k \geq 0$ and hence

$$b_k \geq \max_{0 \leq t_1 < t_2 \leq T} \left[\sum_{\tau=t_1+1}^{t_2} (d_k(\tau) - f_{1k}) \right]^+. \quad (34)$$

Thus we get $h \geq \sum_{k \in \mathcal{N}_D} b_k \geq h_{min}$. If $\Pi^{\{1\}}(h)$ is feasible, then $P(h)$ is also feasible and hence $h \geq h_{min}$ is necessary for both problems to be feasible. Now we prove that it is also sufficient. In particular, we show that for $h = h_{min}$, $\Pi^{\{1\}}(h)$ is feasible. For convenience, define

$$\tilde{h}_k := \max_{0 \leq t_1 < t_2 \leq T} \left[\sum_{\tau=t_1+1}^{t_2} (d_k(\tau) - f_{1k}) \right]^+, \quad k \in \mathcal{N}_D. \quad (35)$$

Then $h_{min} = \sum_{k \in \mathcal{N}_D} \tilde{h}_k$. Rearranging (35), we get

$$f_{1k} \geq \max_{0 \leq t_1 < t_2 \leq T} \left(\frac{\sum_{\tau=t_1+1}^{t_2} d_k(\tau) - \tilde{h}_k}{t_2 - t_1} \right). \quad (36)$$

Also, by hypothesis, we have

$$f_{1k} \geq \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d_k(\tau)}{t} \right). \quad (37)$$

Combining equations (36) and (37), we have

$$f_{1k} \geq \max \left\{ \max_{0 \leq t_1 < t_2 \leq T} \left(\frac{\sum_{\tau=t_1+1}^{t_2} d_k(\tau) - \tilde{h}_k}{t_2 - t_1} \right), \max_{t \in [T]} \left(\frac{\sum_{\tau=1}^t d_k(\tau)}{t} \right) \right\}. \quad (38)$$

For each $k \in \mathcal{N}_D$, consider a single generator single load system as follows. Let the demand profile be $d_k(t)$, the capacity of the transmission line be f_{1k} and the total available storage budget be \tilde{h}_k . For this system, the right hand side in (38) coincides with the definition of f_{min} in (16). From Proposition 1, it follows that there is a feasible generation profile (say $g^{(k)}(t)$) and a storage control policy $r_k(t)$ that define a feasible flow over this single generator single load

system and meet the demand. Now, for the star network, construct the generation profile $g_1(t)$

$$g_1(t) = \sum_{k \in \mathcal{N}_D} g^{(k)}(t),$$

and operate the storage units at each node $k \in \mathcal{N}_D$ with the control policy $r_k(t)$ defined above. Also, $r_1(t) = 0$ for all $t \in [T]$. It can be checked that this defines a feasible point for $\Pi^{\{1\}}(h_{min})$.

Next, we prove that $p_*(h_{min}) = \pi_*^{\{1\}}(h_{min})$. Let $b_k^*, k \in \mathcal{N}$ be optimal storage capacities for problem $P(h_{min})$. Then the optimal storage capacities satisfy the following relations:

$$\sum_{k \in \mathcal{N}_D} b_k^* \geq h_{min}, \quad \text{and} \quad b_1^* + \sum_{k \in \mathcal{N}_D} b_k^* \leq h_{min}.$$

where the first one follows from (34) and the second one follows from the constraint on the total available storage capacities. Rearranging the above equations, we get $b_1^* = 0$ and hence $p_*(h_{min}) = \pi_*^{\{1\}}(h_{min})$. This completes the proof of part (a).

To prove part (b) of Proposition 3, we start by showing that

$$p_*(\infty) = \pi_*^{\{1\}}(\infty). \quad (39)$$

Assume $P(\infty)$ is feasible. For $h = \infty$, we drop the variables $b_k, k \in \mathcal{N}$, and consider the problems $P(\infty)$ and $\Pi^{\{1\}}(\infty)$ over the variables $g_1(t), r_k(t), k \in \mathcal{N}$. The variables $p_{1k}(t)$ and $s_k(t)$ are defined accordingly for all $k \in \mathcal{N}$. We argue that the optimal points of $P(\infty)$ lie in a bounded set. Note that $|p_{1k}(t)| = |d_k(t) + r_k(t)| \leq f_{1k}$ and thus the control policies $r_k(t)$ are bounded for all $k \in \mathcal{N}_D$. Also, the cost function $c_1(\cdot)$ is convex and hence its sub-level sets [39] are bounded. From the above arguments and the power-balance at bus 1, the optimal policy $r_1(t)$ is also bounded. Thus, the set of optimal solutions of $P(\infty)$ is a bounded set. Furthermore, this set is also closed since the objective function and the constraints are continuous functions. As in the proof of Lemma 2, associate the function $\sum_{t \in [T]} |r_1(t)|$ with every point in the set of optimal solutions of $P(\infty)$. This is a continuous function on a compact set and hence attains a minimum. Consider the optimum of $P(\infty)$ where this minimum is attained. We prove (39) by showing that $r_1^*(t) = 0$ for all $t \in [T]$ at this optimum.

Assume to the contrary, that $r_1^*(t)$ is non-zero for some $t \in [T]$. Define

$$t_0 := \{t \in [T] \mid r_1(t_0) > 0\} \quad \text{and} \quad t_1 := \min \{t \in [t_0 + 1, T] \mid r_1^*(t) < 0\}.$$

Also, define $\Delta := \min \{r_1^*(t_0), -r_1^*(t_1)\}$ and notice that $\Delta > 0$.

Case 1: $g_1^*(t_0) > g_1^*(t_1) + \Delta$: Construct the modified generation and charging/ discharging profiles $\tilde{g}_1(t), \tilde{r}_1(t)$ that differ from $g_1^*(t), r_1^*(t)$ only at t_0 and t_1 as follows:

$$\begin{aligned}\tilde{g}_1(t_0) &= g_1^*(t_0) - \Delta g, & \tilde{g}_1(t_1) &= g_1^*(t_1) + \Delta g, \\ \tilde{r}_1(t_0) &= r_1^*(t_0) - \Delta g, & \tilde{r}_1(t_1) &= r_1^*(t_1) + \Delta g,\end{aligned}$$

where $\Delta g := \min \{\Delta, g_1^*(t_0)\} > 0$. As in the proof of Lemma 2, this is feasible for $P(\infty)$. Also, by Lemma 1:

$$c_1(\tilde{g}_1(t_0)) + c_1(\tilde{g}_1(t_1)) \leq c_1(g_1^*(t_0)) + c_1(g_1^*(t_1)).$$

The details are omitted for brevity. This feasible point satisfies

$$|\tilde{r}_1(t_0)| + |\tilde{r}_1(t_1)| = r_1^*(t_0) - \Delta - r_1^*(t_1) - \Delta < |r_1^*(t_0)| + |r_1^*(t_1)|, \quad (40)$$

and hence defines an optimal point of $P(\infty)$ with a strictly lower value of the function $\sum_{t \in [T]} |r_1(t)|$. This is a contradiction.

Case 2: $g_1^*(t_0) \leq g_1^*(t_1) + \Delta$: As above we construct modified storage control policies $\tilde{r}_k(t)$ for all $k \in \mathcal{N}$, keeping the generation profile constant to define an optimal point of $P(\infty)$ with a lower value of $\sum_{t \in [T]} |r_1(t)|$ to derive a contradiction.

Let the modified control policy at bus 1 be as follows:

$$\tilde{r}_1(t_0) = r_1^*(t_0) - \Delta, \quad \tilde{r}_1(t_1) = r_1^*(t_1) + \Delta.$$

Instead, we distribute this to storage devices at $k \in \mathcal{N}_D$, as follows:

$$\tilde{r}_k(t_0) = r_k^*(t_0) + \psi_k, \quad \tilde{r}_k(t_1) = r_k^*(t_1) - \psi_k, \quad k \in \mathcal{N}_D,$$

for some $\psi_k \geq 0, k \in \mathcal{N}_D$ and $\sum_{k \in \mathcal{N}_D} \psi_k = \Delta$. To ensure feasibility of the modified profiles it suffices to check that the line flow constraints are satisfied at t_0 and t_1 . In other words, we show that there exists $\psi_k, k \in \mathcal{N}_D$ such that for all $k \in \mathcal{N}_D$,

$$\psi_k \geq 0, \quad p_{1k}^*(t_0) + \psi_k \leq f_{1k}, \quad p_{1k}^*(t_1) - \psi_k \geq -f_{1k}, \quad \sum_{k \in \mathcal{N}_D} \psi_k = \Delta.$$

Equivalently, we prove that

$$\sum_{k \in \mathcal{N}_D} \min \{f_{1k} - p_{1k}^*(t_0), f_{1k} + p_{1k}^*(t_1)\} \geq \Delta.$$

Recall that $p_{1k}^*(t_0)$ and $p_{1k}^*(t_1)$ are feasible for $P(\infty)$. Thus $p_{1k}^*(t_0) \leq f_{1k}$ and $p_{1k}^*(t_1) \geq -f_{1k}$. Also, $g_1^*(t) - r_1^*(t) = \sum_{k \in \mathcal{N}_D} p_{1k}^*(t)$ at $t = t_0$ and $t = t_1$. Thus, we have

$$\begin{aligned} \sum_{k \in \mathcal{N}_D} \min \{f_{1k} - p_{1k}^*(t_0), f_{1k} + p_{1k}^*(t_1)\} &\geq \sum_{k \in \mathcal{N}_D} (p_{1k}^*(t_1) - p_{1k}^*(t_0)) \\ &= \underbrace{g_1^*(t_1) - g_1^*(t_0)}_{\geq -\Delta} - \underbrace{r_1^*(t_1)}_{\leq -\Delta} + \underbrace{r_1^*(t_0)}_{\geq \Delta} \\ &\geq \Delta, \end{aligned}$$

where the last inequality follows from the hypothesis $g_1^*(t_0) \leq g_1^*(t_1) + \Delta$. The modified profiles satisfy $|\tilde{r}_1(t_0)| + |\tilde{r}_1(t_1)| < |r_1^*(t_0)| + |r_1^*(t_1)|$ as in (40). As argued above this is a contradiction and hence (39) holds.

For $P(\infty)$, $s_k^*(t), k \in \mathcal{N}, t \in [T]$ is finite. Define $h_o := \sum_{k \in \mathcal{N}_D} \max_{t \in [T]} s_k^*(t)$. Then, note that $(g_1^*(t), r_k^*(t), t \in [T], k \in \mathcal{N})$ are also feasible for $\Pi^{\{1\}}(h)$ and $P(h)$ for all $h \geq h_o$. This completes the proof of Proposition 3. ■